Let $F : \mathbf{Z}^+ \to \mathbf{Z}^+$ be defined by $F(n) =$ the number of positive integer divisors of n.

SCORE: _____ / 12 PTS

NOTE: F is a function. You do NOT need to prove that.

NO POINTS FOR ANSWERS WITHOUT EXPLANATIONS

[a] Find $F(12)$. **Justify your answer clearly & briefly.**

$F(12) = 6$ $\quad$ 6 positive integer divisors of $12 = 1, 2, 3, 4, 6, 12$ $\quad$ 1

[b] Find $F(\{5, 8\})$. **Justify your answer clearly & briefly.**

$F(\{5, 8\}) = \boxed{\{2, 4\}}$ $\quad$ 1 PROPER SET NOTATION

2 positive integer divisors of $5 = 1, 5$

4 positive integer divisors of $8 = 1, 2, 4, 8$ $\quad$ 1

[c] Find $F^{-1}(\{1\})$. **Justify your answer clearly & briefly.**

$F^{-1}(\{1\}) = \{1\}$ $\quad$ $F(n) = 1 \Rightarrow n$ has only one positive integer divisor $\Rightarrow n = 1$ $\quad$ 1

[d] What is $F^{-1}(\{2\})$ more commonly known as ? **Justify your answer clearly & briefly.**

1 the set of prime numbers

$F(n) = 2 \Rightarrow n$ has only two positive integer divisors (ie. 1 and itself) $\Rightarrow n$ is prime $\quad$ $1\frac{1}{2}$

[e] Determine if F is one-to-one. **If yes, justify your answer clearly & briefly.** **If no, give an explicit counterexample.**

$1\frac{1}{2}$ no, since $F(2) = F(3) = 2$

[f] Determine if F is onto. **If yes, justify your answer clearly & briefly.** **If no, give an explicit counterexample.**

1 yes – positive integer divisors of $2^{y-1} = 1\ (= 2^0), 2\ (= 2^1), 4\ (= 2^2), \ldots, 2^{y-1}$ $\Rightarrow$ $F(2^{y-1}) = y$ for every $y \in \mathbf{Z}^+$

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[g] Determine if $F^{-1}(2)$ exists. **If yes, find its value.** **If no, explain briefly why not.**

1 no, since F is not one-to-one

Let $S = \{1, 2, 3\}$. Let $F \subseteq \wp(S) \times \wp(S)$ be defined by $(X, Y) \in F$ if and only if $X \cup Y = S$. **SCORE: _____ / 5 PTS**

[a] Determine if $\{1, 3\} F \{2\}$. **Justify your answer clearly & briefly.**

yes, since $\{1, 3\} \cup \{2\} = \{1, 2, 3\}$ $1\frac{1}{2}$

[b] Determine if F is a well-defined function. If yes, find $F(\{3\})$. If no, give an explicit counterexample.

$1\frac{1}{2}$ 1

1 no, since $\{1, 3\} \cup \{2, 3\} = \{1, 2, 3\}$, so $\{1, 3\} F \{2, 3\}$ but $\{2\} \neq \{2, 3\}$, violating uniqueness requirement of a function

Prove that $F : \mathbf{Z}\times\mathbf{Z} \to \mathbf{Z}\times\mathbf{Z}$ defined by $F(x, y) = (x - y, x - 2y)$ is a one-to-one correspondence. **SCORE: _____ / 10 PTS**

Let $(x, y), (a, b)$ be particular but arbitrarily chosen elements of $\mathbf{Z}\times\mathbf{Z}$ such that $F(x, y) = F(a, b)$ 1

So, $(x - y, x - 2y) = (a - b, a - 2b)$

So, $x - y = a - b$ and $x - 2y = a - 2b$ 1

So, $x - y - (x - 2y) = a - b - (a - 2b)$ 1

$\frac{1}{2}$ So, $y = b$

$\frac{1}{2}$ So, $x - b = a - b$

$\frac{1}{2}$ So, $x = a$

So, $(x, y) = (a, b)$ 1

Therefore, F is one-to-one by definition of one-to-one $\frac{1}{2}$

Let (a, b) be a particular but arbitrarily chosen elements of $\mathbf{Z}\times\mathbf{Z}$ 1

1 $F(2a - b, a - b) = (2a - b - (a - b), 2a - b - 2(a - b)) = (a, b)$

where $(2a - b, a - b) \in \mathbf{Z}\times\mathbf{Z}$ by closure of $\mathbf{Z}$ under $\cdot, +$ 1

Therefore, F is onto by definition of onto $\frac{1}{2}$

$x - y = a$ and $x - 2y = b$

$\Rightarrow \quad x - y - (x - 2y) = a - b$

$\Rightarrow \quad y = a - b$

$\Rightarrow \quad x - (a - b) = a$

$\Rightarrow \quad x = 2a - b$

Therefore, F is a one-to-one correspondence by definition of one-to-one correspondence $\frac{1}{2}$